How to use the negation map in the Pollard rho method

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Joint work with Daniel J. Bernstein and Tanja Lange

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A few words about Taiwan and Academia Sinica

- Taiwan (台灣) is an island south of China
- About 36,200 km^2 large
- Territory of the Republic of China (not to be confused with the People's Republic of China)
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- Academia Sinica is a research facility funded by ROC
- About 30 institutes
- More than 800 principal investigators, about 900 postdocs and more than 2200 students

A picture from Taiwan – Sun-Moon Lake (日月潭)

For more pictures check out http://cryptojedi.org/gallery/

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$$k \cdot P = Q$$
- For certain groups G this problem is the basis of many asymmetric cryptographic protocols
- Most importantly: $\mathbb{Z}/n\mathbb{Z}$ and elliptic-curve groups

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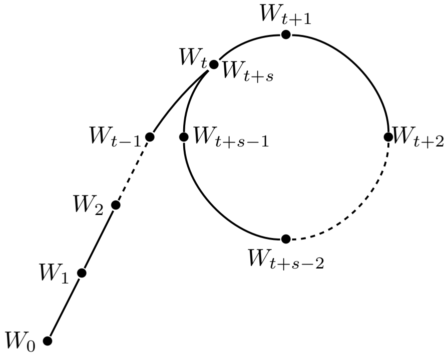
- f needs to preserve knowledge about the linear combination in P and Q
- If $W_i = W_j$ for $i \neq j$, then

$$n_i P + m_i Q = n_j P + m_j Q \Rightarrow$$
$$k = (n_j - n_i)/(m_i - m_j) \mod |G|$$

Pollard's rho algorithm II



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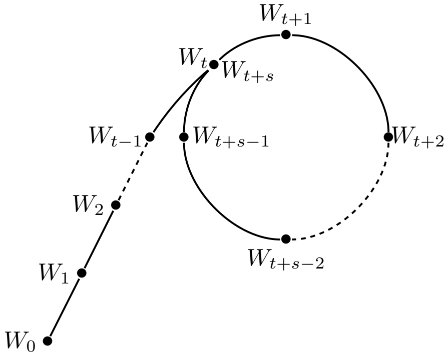

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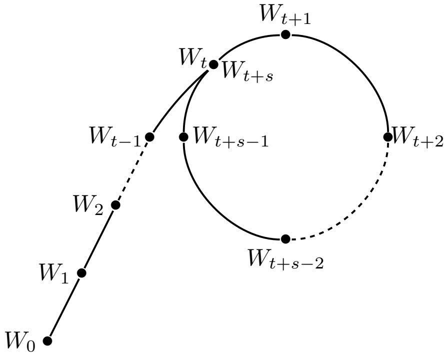

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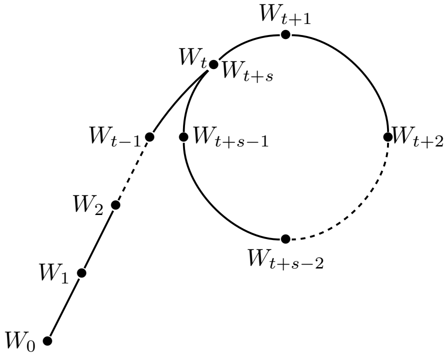

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- Detect cycles without storing all W_i: Floyd, Brent

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- Server searches in incoming points for collisions (same DP, different starting point)

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- Choice of DP-property influences length of separate walks
- Fewer DPs: longer walks (on average), less storage, less communication
- More DPs: Less overhead after a collision
- Clients do not have to update n_i and m_i, simply do successful walks again to find coefficients

Additive walks

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- Precompute r pseudorandom elements $R_0, \ldots, R_{r-1}$ with known linear combination in P and Q
- Use hash function $h : G \to \{0, r-1\}$
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- Summary: additive walks provide much better performance in practice

Application to elliptic-curve groups

- So far, everything worked in the generic-group model
- Now consider groups of points on elliptic curves
- Group elements are points (x, y)
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- Idea: Define iterations on equivalence classes modulo negation
- For example: always take the lexicographic minimum of $(x, -y)$ and (x, y)
- This halves the size of the search space, expected number of iterations drops by a factor of $\sqrt{2}$

Putting it together

- Precompute $R_0, \ldots, R_{r-1}$
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- Iteratively compute $W_{i+1} = |W_i + R_{h(W_i)}|$
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- Probability for such fruitless cycles: $1/2r$
- Similar observations hold for longer fruitless cycles (length 4,6,. . .)
- Probability of a cycle of length $2c$ is $\approx 1/r^c$

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Escape strategies

- Retroactively adjust $h(W_i)$
- Determine unique point in cycle, add “special point” to escape
- Determine unique point in cycle, double this point
- Important: Escape point must be independent of the entrance point

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  "If the Pollard rho method is parallelized in SIMD fashion, it is a challenge to achieve any speedup at all. . . . Dealing with cycles entails administrative overhead and branching, which cause a non-negligible slowdown when running multiple walks in SIMD-parallel fashion. . . . [This] is a major obstacle to the negation map in SIMD environments."

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- Question: Can we really not get the factor-$\sqrt{2}$ speedup with SIMD?

Our approach

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- With even lower frequency check for 4-cycles, 6-cycles etc.
- Implementation actually checks for 12-cycles (with very low frequency)

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- Selection bit is output of branchfree comparison between W_{i-1} and W_{i-1-c} when detecting c-cycles
- All selections, subtractions, additions and comparisons are linear-time
- Asymptotically negligible compared to finite-field multiplications in EC arithmetic

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- Negligible if $r \to \infty$ as $p \to \infty$

Solving the 112-bit ECDLP faster

- Software by Bos et al. takes expected 65.16 PS3 years to solve DLP
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- Paper is online, e.g. at http://cryptojedi.org/papers/#negation