Engineering Cryptographic Software

Scalar multiplication

Radboud University, Nijmegen, The Netherlands

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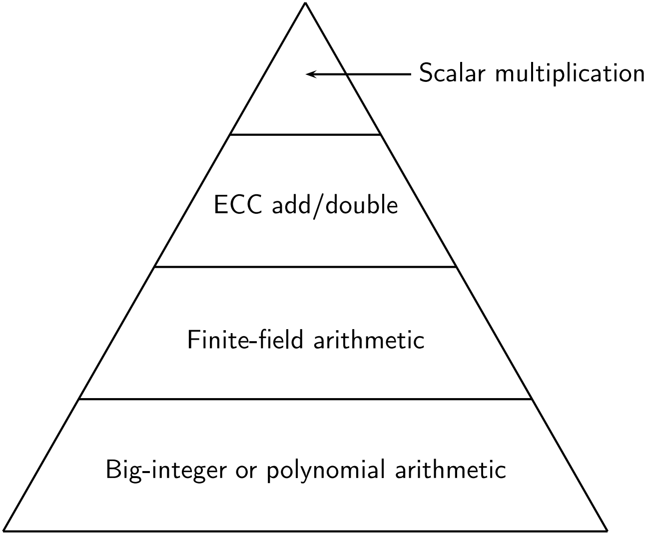
"Today I will be your substitute teacher"

- **Amber Sprenkels**
  Office: Mercator I, 3.11B
  amber@electricdusk.com
  (she/they)

The ECC pyramid



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  - Compute $k \cdot P$ for $k \in \mathbb{Z}$ and $P \in G$

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- Setting for this lecture (peak of the pyramid):
  - Consider (finite, abelian) group G, written additively
  - Compute $k \cdot P$ for $k \in \mathbb{Z}$ and $P \in G$
  - This is the same as x^k for x in a multiplicative group G'
  - Same algorithms for scalar multiplication and exponentiation

ECC in one minute

- (Peter will explain this in more depth in the coming weeks)

- The are points $P, Q, \ldots$
- All the points on an elliptic curve form a *group*
  - You can add/subtract points together, i.e. $R := P + Q$ where $P \neq Q$, or
  - You can double points, i.e. $R := [2]P$
  - This is done using two formulas resp. (called *addition formulas*)
  - There is a *neutral element* $\mathcal{O}$ (like the number 0), i.e. $P + \mathcal{O} = P$
- You cannot multiply a point with another point, i.e., $P \cdot Q$ is invalid
- You can multiply a point with a scalar, i.e., kP (or $[k]P$)

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- Scalar arithmetic is faster than point arithmetic
- More concretely: point addition uses roughly 10 – 50 field operations

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- Typical setting for cryptosystems:
  - P is a fixed system parameter,
  - k is the secret (private) key,
  - Q is the public key.
- Key generation needs to compute $Q = kP$, given k and P

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- Alice computes joint key as $K = k_A Q_B$
- Bob computes joint key as $K = k_B Q_A$

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- Verify: compute $\overline{R} = SP + H(R, M)Q_A$ and check that

$$H(\overline{R}, M) \stackrel{?}{=} H(R, M)$$

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- In the following: Distinguish these cases

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- Conclusion: we need algorithms that run in polynomial time (in the size of the scalar)

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- Cost: 6 doublings, 3 additions
- General algorithm: "Double and add"

  $R \leftarrow P$
  for $i \leftarrow n - 2$ downto 0 **do**
      $R \leftarrow 2R$
      if $(k)_2[i] = 1$ **then**
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- Running time clearly depends on the scalar: insecure for secret scalars!

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- We can do better ($\mathcal{O}$ denotes the neutral element):

  $R \leftarrow \mathcal{O}$
  for $i \leftarrow \max(n_1, n_2) - 1$ downto 0 **do**
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- $\max(n_1, n_2)$ doublings, $m_1 + m_2$ additions

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- Modified algorithm (special case of Strauss' algorithm):

$R \leftarrow \mathcal{O}$
for $i \leftarrow \max(n_1, n_2) - 1$ downto 0 **do**
  $R \leftarrow 2R$
  if $(k_1)_2[i] = 1$ AND $(k_2)_2[i] = 1$ **then**
    $R \leftarrow R + T$
  else if $(k_1)_2[i] = 1$ **then**
    $R \leftarrow R + P_1$
  else if $(k_2)_2[i] = 1$ **then**
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- Modified scalar-multiplication algorithm:

  $R \leftarrow \mathcal{O}$
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- Eliminated all doublings in fixed-basepoint scalar multiplication!

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- Still not constant time, more later. . .

Let's rewrite that a bit . . .

- We have a table $T = (\mathcal{O}, P)$
- Notation $T[0] = \mathcal{O}$, $T[1] = P$
- Scalar multiplication is

  $R \leftarrow P$
  for $i \leftarrow n - 2$ downto 0 **do**
  $\quad R \leftarrow 2R$
  $\quad R \leftarrow R + T[(k)_2[i]]$
  end for

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- How about some nice numbers, like $4, 8, 16$?

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- This is the same as chopping the binary scalar into "windows" of fixed length w
- Compute scalar multiplication as

  $R \leftarrow T[(k)_{2^w}[m-1]]$
  for $i \leftarrow m-2$ downto 0 **do**
      for $j \leftarrow 1$ to w **do**
          $R \leftarrow 2R$
      end for
      $R \leftarrow R + T[(k)_{2^w}[i]]$
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- Larger w: More precomputation
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- For ≈ 256-bit scalars choose $w = 4$ or $w = 5$

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  - Remember that table lookups are generally not constant time!

Making it constant time

```
/* Sets r to the neutral element on the elliptic curve */
extern ec_point_setneutral(ec_point *r);

/* Adds p and q and stores the result in r */
extern ec_point_add(ec_point *r, const ec_point *p, const ec_point *q);

/* Doubles p and stores the result in r */
extern ec_point_double(ec_point *r, const ec_point *p);

/* For point P contains pre-computed multiples P, 2*P, 3*P,...,255*P */
extern ec_point precomputed[255];

ec_scalarmult_P(unsigned char scalar[32])
{
  int i,j;
  ec_point r;
  ec_setneutral(&r);

  for(i=31;i>=0;i--)
  {
    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    if(scalar[i] != 0)
      ec_point_add(&r,&r,precomputed[scalar[i]-1]);
  }
}
```

Making it constant time

```
/* Sets r to the neutral element on the elliptic curve */
extern ec_point_setneutral(ec_point *r);

/* Adds p and q and stores the result in r */
extern ec_point_add(ec_point *r, const ec_point *p, const ec_point *q);

/* Doubles p and stores the result in r */
extern ec_point_double(ec_point *r, const ec_point *p);

/* For point P contains pre-computed multiples 0, P, 2*P, 3*P,...,255*P */
extern ec_point precomputed[256];

ec_scalarmult_P(unsigned char scalar[32])
{
  int i,j;
  ec_point r;
  ec_setneutral(&r);

  for(i=31;i>=0;i--)
  {
    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    ec_point_add(&r,&r,precomputed[scalar[i]]);
  }
}
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/* For point P contains pre-computed multiples 0, P, 2*P, 3*P,...,255*P */
extern ec_point precomputed[256];

ec_scalarmult_P(unsigned char scalar[32])
{
  int i,j;
  ec_point r,t;
  ec_setneutral(&r);

  for(i=31;i>=0;i--)
  {
    for(j=0;j<8;j++)
      ec_point_double(&r,&r);
    ec_point_lookup(&t,precomputed,scalar[i]);
    ec_point_add(&r,&r,&t);
  }
}
```

ec_point_lookup

```
static void ec_point_lookup(ec_point *t, const ec_point *table, int pos)
{
  int i,j;
  unsigned char b;
  *t = table[0];
  for(i=0;i<256;i++)
  {
    b = (i == pos); // Not constant time!
    ec_point_cmov(t, table[i], b); // Copy table[i] to t if b is 1
  }
}
```

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```
static void ec_point_lookup(ec_point *t, const ec_point *table, int pos)
{
  int i,j;
  unsigned char b;
  *t = table[0];
  for(i=0;i<256;i++)
  {
    b = int_eq(i, pos); // set b=1 if i==pos, else set b=0
    ec_point_cmov(t, table[i], b); // Copy table[i] to t if b is 1
  }
}
```

int_eq and ec_point_cmov

```
unsigned char int_eq(int a, int b)
{
  unsigned long long t = a ^ b;
  t = (-t) >> 63;
  return 1-t;
}
```

```
void ec_point_cmov(ec_point *r, const ec_point *t, unsigned char b)
{
  unsigned char *u = (unsigned char *)r;
  unsigned char *v = (unsigned char *)t;
  int i;
  b = -b;
  for(i=0;i<sizeof(ec_point);i++)
    u[i] = (b & v[i]) ^ (~b & u[i]);
}
```

More offline precomputation

- Let's get back to fixed-basepoint multiplication
- So far we precomputed $P, 2P, 4P, 8P, \ldots$

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- Precompute $T_i = (\mathcal{O}, P, 2P, 3P, \dots, (2^w - 1)P) \cdot 2^i$ for $i = 0, w, 2w, 3w, \lceil n/w \rceil - 1$

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- Perform scalar multiplication as

  $R \leftarrow T_0[(k)_{2^w}[0]]$
  for $i \leftarrow 1$ to $\lceil n/w \rceil - 1$ **do**
  $\quad R \leftarrow R + T_{iw}[(k)_{2^w}[i]]$
  end for

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  end for
- No doublings, only $\lceil n/w \rceil - 1$ additions
- Can use huge w, but:
  - at some point the precomputed tables don't fit into cache anymore.
  - constant-time loads get slow for large w

Fixed-window limitations

- Consider the scalar $22 = (1\,01\,10)_2$ and window size 2
  - Initialize R with P
  - Double, double, add P
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- Problem with fixed window: it's fixed.
- Idea: "slide" the window over the scalar

Sliding window scalar multiplication

- Choose window size w
- Rewrite scalar k as $k = (k_0, \ldots, k_m)$ with k_i in $\{0, 1, 3, 5, \ldots, 2^w - 1\}$ with at most one non-zero entry in each window of length w

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- Do this by scanning k from right to left, expand window from each 1-bit
- Precompute $P, 3P, 5P, \ldots, (2^w - 1)P$
- Perform scalar multiplication

  $R \leftarrow \mathcal{O}$
  for $i \leftarrow m$ to 0 **do**
      $R \leftarrow 2R$
      if $k_i \neq 0$ **then**
          $R \leftarrow R + k_iP$
      end if
  end for

Analysis of sliding window

- We still do $n-1$ doublings for an n-bit scalar
- Precomputation needs $2^{w-1}-1$ additions
- Expected number of additions in the main loop: $n/(w+1)$

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- Expected number of additions in the main loop: $n/(w+1)$
- For the same w only half the precomputation compared to fixed-window scalar multiplication
- For the same w fewer additions in the main loop
- But: It's not running in constant time!
- Still nice (in double-scalar version) for signature verification

Differential addition

- Consider elliptic curves of the form $By^2 = x^3 + Ax^2 + x$.
- Montgomery in 1987 showed how to perform x-coordinate-based arithmetic:
  - Given the x-coordinate x_P of P, and
  - given the x-coordinate x_Q of Q, and
  - given the x-coordinate x_{P-Q} of $P - Q$

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  - compute the x-coordinate x_R of $R = P + Q$
- This is called *differential addition*
- Less efficient differential-addition formulas for other curve shapes
- Can be used for efficient computation of the x-coordinate of kP given only the x-coordinate of P
- For this, let's use projective representation $(X : Z)$ with $x = (X/Z)$

One Montgomery "ladder step"

const $a24 = (A+2)/4$ (A from the curve equation)
function ladderstep($X_{Q-P}, X_P, Z_P, X_Q, Z_Q$)
$t_1 \leftarrow X_P + Z_P$
$t_6 \leftarrow t_1^2$
$t_2 \leftarrow X_P - Z_P$
$t_7 \leftarrow t_2^2$
$t_5 \leftarrow t_6 - t_7$
$t_3 \leftarrow X_Q + Z_Q$
$t_4 \leftarrow X_Q - Z_Q$
$t_8 \leftarrow t_4 \cdot t_1$
$t_9 \leftarrow t_3 \cdot t_2$
$X_{P+Q} \leftarrow (t_8 + t_9)^2$
$Z_{P+Q} \leftarrow X_{Q-P} \cdot (t_8 - t_9)^2$
$X_{2P} \leftarrow t_6 \cdot t_7$
$Z_{2P} \leftarrow t_5 \cdot (t_7 + a24 \cdot t_5)$
return $(X_{2P}, Z_{2P}, X_{P+Q}, Z_{P+Q})$
end function

The Montgomery ladder

Require: A scalar $0 \leq k \in \mathbb{Z}$ and the x-coordinate x_P of some point P
Ensure: (X_{kP}, Z_{kP}) fulfilling $x_{kP} = X_{kP}/Z_{kP}$

```
X_1 = x_P; X_2 = 1; Z_2 = 0; X_3 = x_P; Z_3 = 1
for i ← n − 1 downto 0 do
    if bit i of k is 1 then
        (X3, Z3, X2, Z2) ← ladderstep(X1, X3, Z3, X2, Z2)
    else
        (X2, Z2, X3, Z3) ← ladderstep(X1, X2, Z2, X3, Z3)
    end if
end for
return X_2/Z_2
```

The Montgomery ladder (ctd.)

Require: A scalar $0 \leq k \in \mathbb{Z}$ and the x-coordinate x_P of some point P
Ensure: (X_{kP}, Z_{kP}) fulfilling $x_{kP} = X_{kP}/Z_{kP}$

$X_1 = x_P$; $X_2 = 1$; $Z_2 = 0$; $X_3 = x_P$; $Z_3 = 1$
for $i \leftarrow n-1$ downto 0 **do**
  $b \leftarrow$ bit i of s
  $c \leftarrow b \oplus p$
  $p \leftarrow b$
  $(X2, X3) \leftarrow$ cswap$(X2, X3, c)$
  $(Z2, Z3) \leftarrow$ cswap$(Z2, Z3, c)$
  $(X2, Z2, X3, Z3) \leftarrow$ ladderstep$(X1, X2, Z2, X3, Z3)$
end for
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Advantages of the Montgomery ladder

- Very regular structure, easy to protect against timing attacks
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- Very fast (at least if we don't compare to curves with efficient endomorphisms)
- Point compression/decompression is free
- Easy to implement
- No ugly special cases (see Bernstein's "Curve25519" paper)

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- We looked at $n = 2$ before, how about $n = 128$?

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- Assume $k_1 > k_2 > \cdots > k_n$.
- Use that $k_1 P_1 + k_2 P_2 = (k_1 - k_2) P_1 + k_2 (P_1 + P_2)$
- Replace:
  - $(k_1 P_1)$ and $(k_2 P_2)$, with
  - $(k_1 - k_2) P_1$ and $k_2 (P_1 + P_2)$
- Each step requires one scalar subtraction and one point addition
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- Can be very fast (but not constant-time)

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- Each step requires one scalar subtraction and one point addition
- Each step "eliminates" expected $\mathcal{O}(\log n)$ scalar bits
- Can be very fast (but not constant-time)
- Requires fast access to the two largest scalars: put scalars into a heap
- Crucial for good performance: fast heap implementation

A fast heap

- Heap is a binary tree, each parent node is larger than the two child nodes
- Data structure is stored as a simple array, positions in the array determine positions in the tree
- Root is at position 0, left child node at position 1, right child node at position 2 etc.
- For node at position i, child nodes are at position $2 \cdot i + 1$ and $2 \cdot i + 2$, parent node is at position $\lfloor (i-1)/2 \rfloor$

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- Typical heap root replacement (pop operation): start at the root, swap down for a variable amount of times
- Floyd's heap: swap down to the bottom, swap up for a variable amount of times, advantages:
  - Each swap-down step needs only one comparison (instead of two)
  - Swap-down loop is more friendly to branch predictors

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- Optimizing for the scalar means that the scalar has to be public
- Not the typical setting for ECC
- Some applications:
  - Inversion in finite fields (cmp. slides 55&56 of multiprecision.pdf)
  - Elliptic-curve factorization method (not in this lecture)

Addition chains

Definition

Let k be a positive integer. A sequence $s_1, s_2, \ldots, s_m$ is called an addition chain of length m for k if

- $s_1 = 1$
- $s_m = k$
- for each s_i with $i > 1$ it holds that $s_i = s_j + s_\ell$ for some $j, \ell < i$

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- Signed-scalar representations are "addition-subtraction chains"
- For fixed scalar we can spend a lot of time to find a good addition chain at compile time
- This is what will be used for inversion in $\mathbb{F}_{2^{255}-19}$
- Computing good addition chains? See github.com/mmcloughlin/addchain