Cryptographic Engineering
Multiprecision arithmetic II and ECC

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IN·DEI·NOMINE·FELICITER

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Where were we...?

- Last lecture: arithmetic on big integers
- Conclusion at the end:
  - Can use a *redundant representation* for big integers
  - Carries get accumulated in "unused" upper parts of registers
  - Arithmetic becomes essentially polynomial arithmetic
  - Need to carry en bloc whenever coefficients become too large

Example: product-scanning multiplication

```
/* 256-bit integers in radix 2^16 */
typedef signed long long bigint[16];

void mul_prodscan(signed long long r[31],
                  const bigint x,
                  const bigint y)
{
  r[0]   = x[0] * y[0];
  r[1]   = x[1] * y[0];
  r[1]  += x[0] * y[1];
  r[2]   = x[2] * y[0];
  r[2]  += x[1] * y[1];
  r[2]  += x[0] * y[2];
  ...
  r[29]  = x[15] * y[14];
  r[29] += x[14] * y[15];
  r[30]  = x[15] * y[15];
}
```

Modular reduction

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- We know that $2^{255} \equiv 19 \pmod{p}$
- This means that $2^{256} \equiv 38 \pmod{p}$
- Reduce 31-bit intermediate result `r` as follows:

```
for(i=0;i<15;i++)
  r[i] += 38*r[i+16];
```

- Result is in `r[0]`,...,`r[15]`

Primes are not rabbits

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  - $2^{192} - 2^{64} - 1$ ("NIST-P192", FIPS186-2, 2000)
  - $2^{224} - 2^{96} + 1$ ("NIST-P224", FIPS186-2, 2000)
  - $2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$ ("NIST-P256", FIPS186-2, 2000)
  - $2^{255} - 19$ (Bernstein, 2006)
  - $2^{251} - 9$ (Bernstein, Hamburg, Krasnova, Lange, 2013)
  - $2^{448} - 2^{224} - 1$ (Hamburg, 2015)

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- All these primes come with (more or less) fast reduction algorithms
- More about *general primes* later
- For the moment let's stick to $2^{255} - 19$

Carrying after multiplication

```
long long c;
for(i=0;i<15;i++)
{
  c = r[i] >> 16;
  r[i+1] += c;
  c <<= 16;
  r[i] -= c;
}
c = r[15] >> 16;
r[0] += 38*c;
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r[15] -= c;
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- Coefficient `r[0]` may still be too large: carry again to `r[1]`

How about squaring?

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#define bigint_square(R,X) bigint_mul(R,X,X)
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/* 256-bit integers in radix 2^16 */
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void square_prodscan(signed long long r[31],
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  signed long long _2x[16];
  int i;
  for(i=0;i<16;i++)
    _2x[i] = 2*x[i];

  r[0]   =   x[0] * x[0];
  r[1]   = _2x[1] * x[0];
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Squaring vs. multiplication

Multiplication needs

- 256 multiplications
- 225 additions

Squaring needs

- 136 multiplications
- 105 additions
- 15 additions or shifts or multiplications by 2 for precomputation

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- Example: German BSI is pushing the "Brainpool curves", over fields $\mathbb{F}_p$ with

$$p_{224} = 2272162293245435278755253799591092807334073\backslash 2145944992304435472941311$$
$$= 0xD7C134AA264366862A18302575D1D787B09F07579\backslash 7DA89F57EC8C0FF$$

or

$$p_{256} = 7688495639704534422080974662900164909303795\backslash 0200943055203735601445031516197751$$
$$= 0xA9FB57DBA1EEA9BC3E660A909D838D726E3BF623D\backslash 52620282013481D1F6E5377$$

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- Another example: Pairing-friendly curves are typically defined over fields $\mathbb{F}_p$ where p has *some* structure, but hard to exploit for fast arithmetic

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- Better idea (Montgomery, 1985):
  - Let R be such that $\gcd(R, p) = 1$ and $t < p \cdot R$
  - Represent an element a of $\mathbb{F}_p$ as $aR \mod p$
  - Multiplication of aR and bR yields $t = abR^2$ ($2n$ limbs)
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  - For *some* choices of R this is more efficient than division
  - Typical choice for radix-b representation: $R = b^n$

Montgomery reduction (pseudocode)

Require: $p = (p_{n-1}, \ldots, p_0)_b$ with $\gcd(p, b) = 1$, $R = b^n$,
$p' = -p^{-1} \mod b$ and $t = (t_{2n-1}, \ldots, t_0)_b$
Ensure: $tR^{-1} \mod p$

```
A ← t
for i from 0 to n − 1 do
    u ← a_i p' mod b
    A ← A + u · p · b^i
end for
A ← A / b^n
if A ≥ p then
    A ← A − p
end if
return A
```

Some notes about Montgomery reduction

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- One can merge schoolbook multiplication with Montgomery reduction: "Montgomery multiplication"

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- Efficient ECC arithmetic avoids frequent inversions
- ECC can typically not avoid *all* inversions
- We need inversion, but we do (usually) not need it often
- Two approaches to inversion:
  1. Extended Euclidean algorithm
  2. Fermat's little theorem

Extended Euclidean algorithm

- Given two integers a, b, the Extended Euclidean algorithm finds
  - The greatest common divisor of a and b
  - Integers u and v, such that $a \cdot u + b \cdot v = \gcd(a, b)$

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- To compute $a^{-1} \pmod{p}$, use the algorithm to compute

$$a \cdot u + p \cdot v = \gcd(a, p) = 1$$

- Now it holds that $u \equiv a^{-1} \pmod{p}$

Extended Euclidean algorithm (pseudocode)

Require: Integers a and b.
Ensure: An integer tuple (u, v, d) satisfying $a \cdot u + b \cdot v = d = \gcd(a, b)$

$u \leftarrow 1$
$v \leftarrow 0$
$d \leftarrow a$
$v_1 \leftarrow 0$
$v_3 \leftarrow b$
while $(v_3 \neq 0)$ **do**
  $q \leftarrow \lfloor \frac{d}{v_3} \rfloor$
  $t_3 \leftarrow d \mod v_3$
  $t_1 \leftarrow u - qv_1$
  $u \leftarrow v_1$
  $d \leftarrow v_3$
  $v_1 \leftarrow t_1$
  $v_3 \leftarrow t_3$
end while
$v \leftarrow \frac{d-au}{b}$
return (u, v, d)

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- Core operation are divisions with remainder
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- Version without divisions: **binary extended gcd**:

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- The running time (number of loop iterations) depends on the inputs
- We usually do not want this for cryptography (timing attacks!)
- Possible protection: blinding
  - Multiply a by random integer r
  - Invert, obtain $r^{-1}a^{-1}$
  - Multiply again by r to obtain a^{-1}
- Note that this requires a source of randomness

Fermat's little theorem

Theorem

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- Obvious algorithm for inversion: Exponentiation with $p-2$
- The exponent is quite large (e.g., 255 bits), is that efficient?
- Yes, fairly:
  - Exponent is fixed and known at compile time
  - Can spend quite some time on finding an efficient addition chain (next lecture)
  - Inversion modulo $2^{255}-19$ needs 254 squarings and 11 multiplications in $\mathbb{F}_{2^{255}-19}$

Inversion in $\mathbb{F}_{2^{255}-19}$

```
void gfe_invert(gfe r, const gfe x)
{
gfe z2, z9, z11, z2_5_0, z2_10_0, z2_20_0, z2_50_0, z2_100_0, t;
int i;
/* 2 */                gfe_square(z2,x);
/* 4 */                gfe_square(t,z2);
/* 8 */                gfe_square(t,t);
/* 9 */                gfe_mul(z9,t,x);
/* 11 */               gfe_mul(z11,z9,z2);
/* 22 */               gfe_square(t,z11);
/* 2^5 - 2^0 = 31 */ gfe_mul(z2_5_0,t,z9);
/* 2^6 - 2^1 */        gfe_square(t,z2_5_0);
/* 2^10 - 2^5 */       for (i = 1;i < 5;i++) { gfe_square(t,t); }
/* 2^10 - 2^0 */       gfe_mul(z2_10_0,t,z2_5_0);
/* 2^11 - 2^1 */       gfe_square(t,z2_10_0);
/* 2^20 - 2^10 */      for (i = 1;i < 10;i++) { gfe_square(t,t); }
/* 2^20 - 2^0 */       gfe_mul(z2_20_0,t,z2_10_0);
/* 2^21 - 2^1 */       gfe_square(t,z2_20_0);
/* 2^40 - 2^20 */      for (i = 1;i < 20;i++) { gfe_square(t,t); }
/* 2^40 - 2^0 */       gfe_mul(t,t,z2_20_0);
```

Inversion in $\mathbb{F}_{2^{255}-19}$

```
/* 2^41 - 2^1 */      gfe_square(t,t);
/* 2^50 - 2^10 */     for (i = 1;i < 10;i++) { gfe_square(t,t); }
/* 2^50 - 2^0 */      gfe_mul(z2_50_0,t,z2_10_0);
/* 2^51 - 2^1 */      gfe_square(t,z2_50_0);
/* 2^100 - 2^50 */    for (i = 1;i < 50;i++) { gfe_square(t,t); }
/* 2^100 - 2^0 */     gfe_mul(z2_100_0,t,z2_50_0);
/* 2^101 - 2^1 */     gfe_square(t,z2_100_0);
/* 2^200 - 2^100 */   for (i = 1;i < 100;i++) { gfe_square(t,t); }
/* 2^200 - 2^0 */     gfe_mul(t,t,z2_100_0);
/* 2^201 - 2^1 */     gfe_square(t,t);
/* 2^250 - 2^50 */    for (i = 1;i < 50;i++) { gfe_square(t,t); }
/* 2^250 - 2^0 */     gfe_mul(t,t,z2_50_0);
/* 2^251 - 2^1 */     gfe_square(t,t);
/* 2^252 - 2^2 */     gfe_square(t,t);
/* 2^253 - 2^3 */     gfe_square(t,t);
/* 2^254 - 2^4 */     gfe_square(t,t);
/* 2^255 - 2^5 */     gfe_square(t,t);
/* 2^255 - 21 */      gfe_mul(r,t,z11);
}
```

Multiprecision libraries

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  - OpenSSL Bignum (http://openssl.org), low-level routines in OpenSSL
  - mp$\mathbb{F}_q$ (http://mpfq.gforge.inria.fr/), a finite-field library (generator)

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- Libraries are not always timing-attack protected
- Consequence: ECC speed records are achieved with hand-optimized assembly implementations

Part II
Elliptic-curve cryptography
from a crypto-engineering perspective

Diffie-Hellman

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Groups with hard DLP

- Traditional answer: $\mathbb{Z}_p^*$ with large prime-order subgroup
- Modern answer: Elliptic curve over $\mathbb{F}_q$ with large prime-order subgroup

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- Bob computes joint key $b(aP)$
- DLP in G: given $kP \in G$ and P, find k
- Solving the DLP breaks security of Diffie-Hellman

Groups with hard DLP

- Traditional answer: $\mathbb{Z}_p^*$ with large prime-order subgroup
- Modern answer: Elliptic curve over $\mathbb{F}_q$ with large prime-order subgroup
- Sophisticated answer (not in this lecture): hyperelliptic curves of genus 2

Diffie-Hellman

- Let G be a cyclic, finite, abelian Group (written additively) and let P be a generator of G
- Alice chooses random $a \in \{0, \ldots, |G| - 1\}$, computes aP, sends to Bob
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Typical view on elliptic curves

Definition

Let K be a field and let $a_1, a_2, a_3, a_4, a_6 \in K$. Then the following equation defines an elliptic curve E:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

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Rational points

Setup for cryptography

- Choose $K = \mathbb{F}_q$
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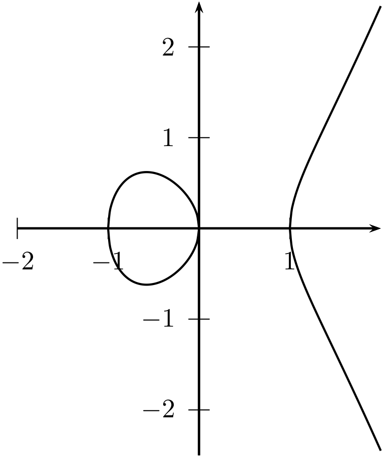
The group law

Example curve: $y^2 = x^3 - x$ over $\mathbb{R}$

Graph of E over $\mathbb{R}$



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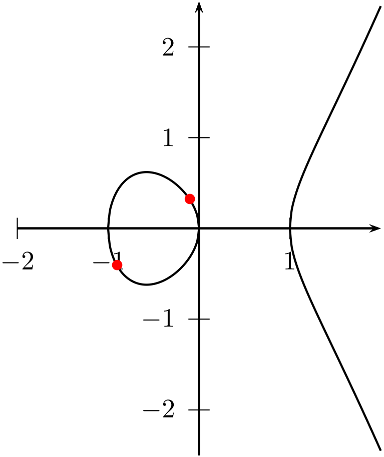

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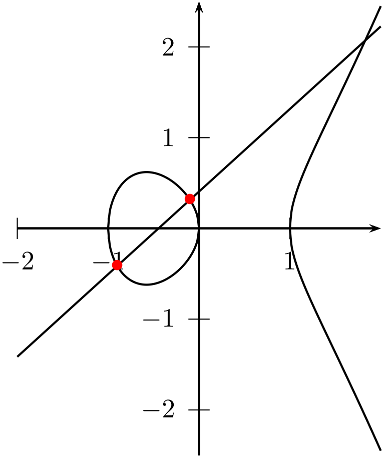

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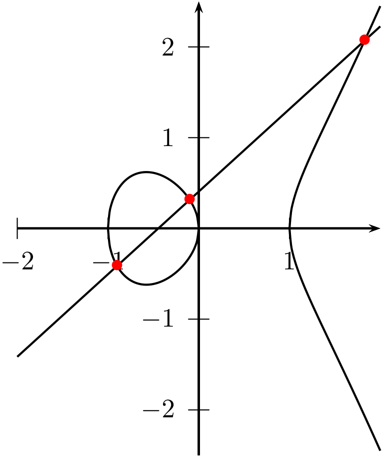

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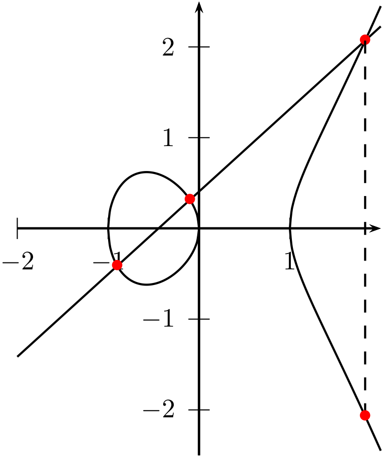

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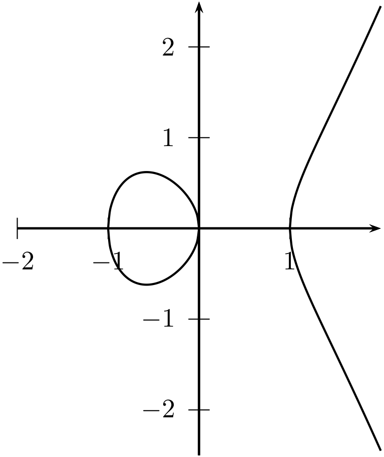

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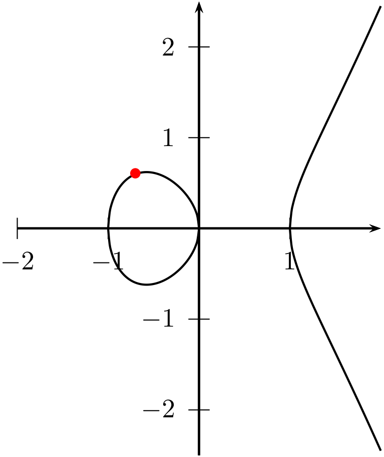

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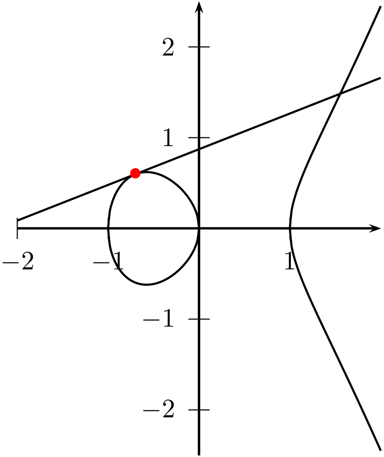

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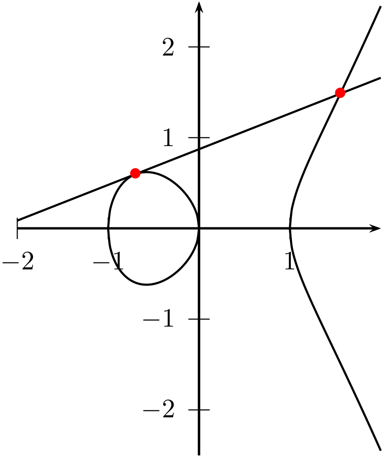

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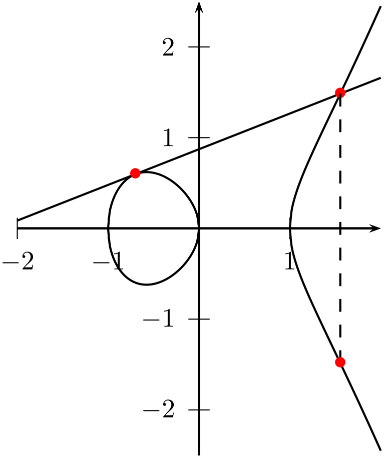

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- Formulas for curves over $\mathbb{F}_{2^k}$ look slightly different, but same special cases

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Finding a curve

- Fix finite field $\mathbb{F}_q$ of suitable size
- Fix curve parameter a (quite common: $a = -3$)
- Pick curve parameter b until E fulfills desired properties
- This requires efficient "point counting"
- This requires efficient factorization or primality proving

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- FRP256v1 (ANSSI), one prime-field curve (256 bits)

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Curves over big-prime fields

- Many fields of a given size $\Rightarrow$ many curves
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Curves over binary fields

- Important for security: exponent k in $\mathbb{F}_{p^k}$ has to be prime
- Not many fields (not that many curves)
- More efficient in hardware
- Efficient in software only on some microarchitectures
- A hell to implement securely in software on some other microarchitectures

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- Important: Never *send* projective representation, always convert to affine!

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- Addition of $P + Q$ needs to distinguish different cases:
  - If $P = \mathcal{O}$ return Q
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- More bad news: Doesn't work for multi-scalar multiplication (next lecture)

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  - Else if $P = -Q$ return $\mathcal{O}$
  - Else use addition formulas
- Similar for doubling P:
  - If $P = \mathcal{O}$ return P
  - Else if $y_P = 0$ return $\mathcal{O}$
  - Else use doubling formulas
- Constant-time implementations of this are horrible
- Good news: Can avoid the checks when computing $k \cdot P$ and $k < |E(\mathbb{F}_q)|$
- Bad news: Side-channel countermeasures use $k > |E(\mathbb{F}_q)|$
- More bad news: Doesn't work for multi-scalar multiplication (next lecture)
- Baseline: *simple* implementations are likely to be wrong or insecure

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  - We only get the x coordinate of the result, tricky for signatures
  - Can reconstruct y, but that involves some additional cost

Solution II: (twisted) Edwards curves

- Edwards, 2007: New form for elliptic curves (“Edwards curves”)
- Bernstein, Lange, 2007: very fast addition and doubling on these curves
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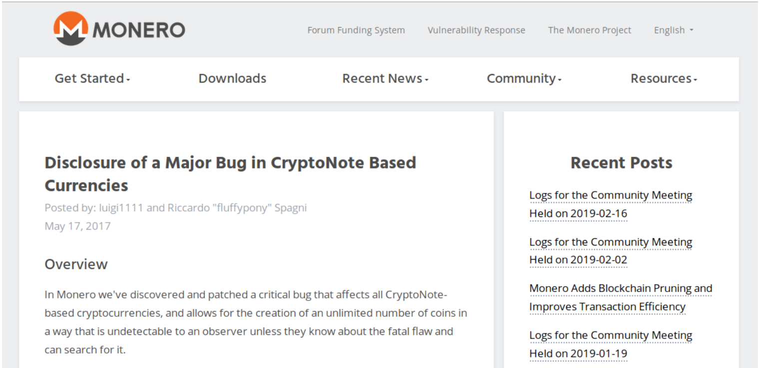
- Disadvantage: always have a cofactor of at least 4

So, what's the deal with the cofactor?



So, what's the deal with the cofactor?

- Protocols need to be careful to avoid subgroup attacks
- Monero screwed this up, which allowed double-spending
- Elegant solution: "Ristretto" encoding by Hamburg, see: https://github.com/otrv4/libgoldilocks

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- Problem: Extremely inefficient
- Renes, Costello, Batina, 2016: Much faster complete group law for Weierstrass curves
- Somewhat less efficient than (twisted) Edwards
- Covers all curves

Problem III: Wrong-curve attacks

ECDH attack scenario

- Alice sends point on different (insecure) curve with small subgroup
- Bob computes "shared key" in that small subgroup
- Alice learns "shared key" through brute force
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- Send compressed points $(x, \text{parity}(y))$; decompression returns (x, y) on the curve or fails
- Send only x (Montgomery ladder); but: x could still be on the "twist" of E
- Make sure that the twist is also secure ("twist security")

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- Fact: There is no proof that there are no intentional vulnerabilities in NIST curves
- For more details, see BADA55 elliptic curves

Choosing a safe curve

Overview of various elliptic curves and thorough security analysis by Bernstein and Lange:

https://safecurves.cr.yp.to

(doesn't list cofactor-1 curves, so best to combine with Ristretto)

Point representation and arithmetic

Collection of elliptic-curve shapes, point representations and group-operation formulas by Bernstein and Lange:

https://www.hyperelliptic.org/EFD/